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On the Coulomb Green function

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Abstract. The connection between the Green function for the isotropic harmonic oscillator in two-dimensional complex space and the Coulomb Green function is established using the method of path integrals. The representations for the Coulomb Green function are proposed to be useful in concrete applications.

1. Introduction

In many applications of analytical methods in perturbation theory the use of the Coulomb Green function plays an important role. In the present paper the connection between the Green function for an isotropic harmonic oscillator in two-dimensional complex space and the Coulomb Green function is established which permits one to obtain a useful representation for the latter. In our calculations we use the connection between the Schrödinger equations which has been found by Komarov and Romanova (1982), since we consider it to be the most adequate (as compared, for example, with the results of Chen (1982), Cornish (1984) and Kibler and Negadi (1983)) for the construction of purely algebraic methods of performing concrete calculations with the Coulomb Green function.

2. The connection with the Green functions

Let us consider the Schrödinger equation in the space with the complex coordinates ξ_s ($s = 1, 2$) (we assume ξ_s to be the spinor components):

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\xi_s^*\,\partial\xi_s}+\tfrac{1}{2}\omega^2\xi_s^*\xi_s\right)\psi(\xi)=z\psi(\xi) \tag{1}$$

where the asterisk denotes the operation of complex conjugation, with summation over repeated indices. In equation (1) ω is a real positive number. The Green function for the equation (1) in the 'energy' representation is a solution of the following equation:

$$\left(z+\frac{1}{2}\frac{\partial^2}{\partial\xi_s^*\,\partial\xi_s}-\tfrac{1}{2}\omega^2\xi_s^*\xi_s\right)U(\xi,\eta;z)=\mathrm{i}\delta(\xi_1'-\eta_1')\delta(\xi_1''-\eta_1'')\delta(\xi_2'-\eta_2')\delta(\xi_2''-\eta_2'') \tag{2}$$

where $\xi_s'=\operatorname{Re}\xi_s$, $\xi_s''=\operatorname{Im}\xi_s$ and $\delta(x)$ is a Dirac δ function. One of the ways of constructing the function $U(\xi,\eta;z)$ is to represent it as a path integral (see, for

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example, Slavnov and Faddeev 1978)

$$U(\xi,\eta;z)=\int_0^\infty d\theta\, e^{iz\theta}\prod_{\theta'} D^4\xi(\theta') \times\exp\left(i\int_0^\theta d\theta'[2\dot\xi_s^*(\theta')\xi_s(\theta')-\tfrac12\omega^2\xi_s^*(\theta')\xi_s(\theta')]\right) \tag{3}$$

where $\dot\xi_s(\theta')=d\xi_s(\theta')/d\theta'$ and $\xi_s(\theta)=\xi_s$, $\xi_s(0)=\eta_s$. Equation (3) is regarded as a limit (when $\varepsilon\to0$, $N\to\infty$, $(N\varepsilon=\theta)$) of the following expression (see Slavnov and Faddeev 1978)

$$U(\xi,\eta;z)=\int_0^\infty d\theta\lim_{\substack{\varepsilon\to0\\N\to\infty}}\left(\frac{2}{i\pi\varepsilon}\right)^{2N}\prod_{k=1}^{N} d^4\xi(k) \times\exp\left[i\sum_{k=1}^{N}\left(\frac{2}{\varepsilon}[\xi_s^*(k)-\xi_s^*(k-1)][\xi_s(k)-\xi_s(k-1)] +z\varepsilon-\tfrac12\omega^2\varepsilon\xi_s^*(k-1)\xi_s(k-1)\right)\right] \tag{4}$$

where $\xi_s(N)=\xi_s$, $\xi_s(0)=\eta_s$ and $d^4\xi(k)=d\xi_1'(k)\,d\xi_1''(k)\,d\xi_2'(k)\,d\xi_2''(k)$ (for brevity, we use the notation $\xi_s(k)\equiv\xi_s(k\varepsilon)$). To establish the relationship sought between the function $U(\xi,\eta;z)$ and the Coulomb Green function, we change in (3) the variables, choosing as part of the new variables the function

$$x_\lambda(\theta)=(\sigma_\lambda)_{st}\xi_s^*(\theta)\xi_t(\theta) \tag{5}$$

where $(\sigma_\lambda)_{st}$ $(\lambda=1,2,3)$ are the matrix elements of the Pauli matrices. In view of the assumed properties of ξ_s the functions $x_\lambda(\theta)$ constitute the components of a three-dimensional vector function. Making use of the relation

$$(\sigma_\lambda)_{st}(\sigma_\lambda)_{uv}=2\delta_{sv}\delta_{tu}-\delta_{st}\delta_{uv} \tag{6}$$

one can easily ascertain that

$$\xi_s^*(\theta)\xi_s(\theta)=\frac{1}{4r(\theta)}\dot x_\lambda(\theta)\dot x_\lambda(\theta)-\frac{1}{4r(\theta)}[\dot\xi_s^*(\theta)\xi_s(\theta)-\xi_s^*(\theta)\dot\xi_s(\theta)]^2 \tag{7}$$

$$\dot x_\lambda(\theta)=\frac{dx_\lambda(\theta)}{d\theta}\qquad r(\theta)=[x_\lambda(\theta)x_\lambda(\theta)]^{1/2}=\xi_s^*(\theta)\xi_s(\theta). \tag{8}$$

From (7) it follows that in calculating the path integral by equation (4), the appropriate substitution of variables is

$$\begin{aligned}&x_\lambda(k)=(\sigma_\lambda)_{st}\xi_s^*(k)\xi_t(k)\\&x_4(k)=i[\xi_s^*(k)-\xi_s^*(k-1)]\xi_s(k)-i\xi_s^*(k)[\xi_s(k)-\xi_s(k-1)]\\&d^4\xi(k)=\frac{1}{16r^2(k)}d^4x(k)\equiv\frac{1}{16r^2(k)}dx_1(k)\,dx_2(k)\,dx_4(k)\end{aligned} \tag{9}$$

$$U(\xi,\eta;z)=\int_0^\infty d\theta\lim_{\substack{\varepsilon\to0\\N\to\infty}}\left(\frac{2}{i\pi\varepsilon}\right)^2\prod_{k=1}^{N-1}\frac{d^4x(k)}{[2i\pi\varepsilon r(k)]^2} \times\exp\left[i\sum_{k=1}^{N}\left(\frac{1}{2\varepsilon r(k)}[x_\lambda(k)-x_\lambda(k-1)][x_\lambda(k)-x_\lambda(k-1)] +\frac{x_4(k)}{2\varepsilon r(k)}+z\varepsilon-\tfrac12\omega^2\varepsilon r(k-1)\right)\right]. \tag{10}$$

The separation of the variables $x_4(k)$ is due to the fact that the function

$$Q = \mathrm{i}[\dot{\xi}_s^*(\theta)\xi_s(\theta) - \xi_s(\theta)\dot{\xi}_s^*(\theta)]$$

is the integral of motion for a system described by the Lagrange function

$$\mathscr{L} = 2\dot{\xi}_s^*(\theta)\dot{\xi}_s(\theta) - \tfrac{1}{2}\omega^2\xi_s(\theta)\xi_s(\theta).$$

Integrating over the variables $x_4(k)$ $(k = 1, 2, \ldots, N-1)$ and taking into account that $U(\xi, \eta; z)$ is the integral operator, which will be used, as shown by Komarov and Romanova (1982), in the class of functions depending on x_λ only (i.e. one may integrate with respect to an 'extra' variable $x_4(N)$), we find

$$\begin{aligned} K(\boldsymbol{r}, \boldsymbol{r}'; -\tfrac{1}{2}\omega^2) &= \frac{1}{16r(N)} \int \mathrm{d}x_4(N)\, U(\xi, \eta; z) \\ &= \int_0^\infty \mathrm{d}\theta \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \frac{r(N)}{(2\mathrm{i}\pi\varepsilon r(N))^{3/2}} \prod_{k=1}^{N} \frac{\mathrm{d}^3 x(k)}{(2\mathrm{i}\pi\varepsilon r(k))^{3/2}} \\ &\quad \times \exp\left[\mathrm{i} \sum_{k=1}^{N} \left(\frac{1}{2\varepsilon r(k)} [x_\lambda(k) - x_\lambda(k-1)][x_\lambda(k) - x_\lambda(k-1)] \right.\right. \\ &\quad \left.\left. + z\varepsilon - \tfrac{1}{2}\omega^2 \varepsilon r(k-1) \right)\right] \end{aligned} \tag{11}$$

where

$$\boldsymbol{r} = \xi_s^*(\boldsymbol{\sigma})_{st}\xi_t \qquad \boldsymbol{r}' = \eta_s^*(\boldsymbol{\sigma})_{st}\eta_t. \tag{12}$$

The weight $(16r(N))^{-1}$ on the integration over $x_4(N)$ is defined by the Jacobian of the transformation (9) and the conditions of normalisation of the wavefunctions (Komarov and Romanova 1982).

The last step is the change of a 'time' variable: let

$$\varepsilon(k) = \varepsilon r(k) \tag{13}$$

which in the limit $\varepsilon \to 0$ is equivalent to the introduction of a new time variable

$$t = \int_0^\theta \mathrm{d}\theta'\, r(\theta'). \tag{14}$$

As a result we have

$$\begin{aligned} K(\boldsymbol{r}, \boldsymbol{r}'; -\tfrac{1}{2}\omega^2) &= \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\frac{1}{2}\omega^2 t} \prod_\tau D^3 x(\tau) \\ &\quad \times \exp\left[\mathrm{i} \int_0^t \mathrm{d}\tau \left(\tfrac{1}{2}\dot{x}_\lambda(\tau)\dot{x}_\lambda(\tau) + \frac{z}{r(\tau)} \right)\right] \end{aligned} \tag{15}$$

where $\boldsymbol{r}(t) = \boldsymbol{r}$ and $\boldsymbol{r}(0) = \boldsymbol{r}'$. Denoting $E = -\frac{1}{2}\omega^2$ we obtain from (15)

$$\left(E + \tfrac{1}{2}\Delta + \frac{z}{r} \right) K(\boldsymbol{r}, \boldsymbol{r}'; E) = \mathrm{i}\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{16}$$

(Δ is the Laplace operator), i.e. $K(\boldsymbol{r}, \boldsymbol{r}'; E)$ is the energy representation of the Coulomb Green function.

A simple method of constructing the Coulomb Green function results from the calculation described. Starting from the well known expression for the Green function of the harmonic oscillator (see, for example, Feynman and Hibbs 1965), we write down the 'energy' representation of the Green function for the isotropic harmonic oscillator in the two-dimensional complex space

$$U(\xi,\eta;z)=-\frac{2\omega}{\pi^2}\int_0^\infty \mathrm{d}t\,\exp\left(\mathrm{i}\frac{2z}{\omega}t\right)(\sin t)^{-2}$$
$$\times\exp\left(\frac{\mathrm{i}\omega}{\sin t}[(\xi_s^*\xi_s+\eta_s^*\eta_s)\cos t-(\xi_s^*\eta_s+\eta_s^*\xi_s)]\right). \tag{17}$$

Changing the variables (12) and choosing as extra variables (see Komarov and Romanova 1982)

$$\chi=\tan^{-1}(\xi_1''/\xi_1') \qquad \chi'=\tan^{-1}(\eta_1''/\eta_1') \tag{18}$$

we arrive at the formula (we use the following spherical coordinates in the spaces $\boldsymbol{r}$ and $\boldsymbol{r}'$: r, θ, φ and r', θ', φ')

$$U(\boldsymbol{r},\chi;\boldsymbol{r}',\chi';z)=-\frac{2\omega}{\pi^2}\int_0^\infty \mathrm{d}t\,\exp\left(\mathrm{i}\frac{2z}{\omega}t\right)(\sin t)^{-2}$$
$$\times\exp\left[\mathrm{i}\omega(r+r')\cot t-2\frac{\mathrm{i}\omega}{\sin t}\sqrt{rr'}\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\cos(\chi-\chi')\right.\right.$$
$$\left.\left.+\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\cos(\chi-\chi'+\varphi-\varphi')\right)\right]. \tag{19}$$

Upon integrating over χ (i.e. over one of the 'extra' variables), we obtain

$$K(\boldsymbol{r},\boldsymbol{r}';E)=\tfrac{1}{8}\int_0^{2\pi}\mathrm{d}\chi\,U(\boldsymbol{r},\chi;\boldsymbol{r}',\chi';z)$$
$$=-\frac{\omega}{2\pi}\int_0^\infty \mathrm{d}t\,\exp\left(\mathrm{i}\frac{2z}{\omega}t\right)(\sin t)^{-2}$$
$$\times\exp[\mathrm{i}\omega(r+r')\cot t]J_0\left(\frac{\omega}{\sin t}\sqrt{2(rr'+\boldsymbol{rr}')}\right) \tag{20}$$

where $J_0(x)$ is the Bessel function (the coefficient $\frac{1}{8}$ is defined by the Jacobian of transformation $\mathrm{d}^4\xi=(1/8r)\,\mathrm{d}^3x\,\mathrm{d}\chi$) It is easy to see (one can change t into $-\mathrm{i}\tau$ and then $\sinh\tau$ into $(\sinh x)^{-1}$) that (20) is one of the forms of the Coulomb Green function in which it was first constructed by Hostler (1964).

3. Example of application: non-relativistic radiation shifts of the levels of hydrogen-like atoms

We shall now show that the relationship established between the Coulomb Green function and the Green function for the isotropic harmonic oscillator in two-dimensional complex space allows one to use the operator form of the function $U(\xi,\eta;z)$ in concrete calculations and thus to reduce rather complicated calculations of the matrix elements of the function $U(\xi,\eta;z)$ to purely algebraic procedures of transforming the products of the creation and annihilation operators to a normal form.

As the specific example, consider the problem of a non-relativistic calculation of shifts and radiative widths of the energy levels of hydrogen-like atoms (in what follows we adhere to the treatment of this problem given by Bjorken and Drell (1964)). The Schrödinger equation for an electron moving in the nuclear Coulomb field with charge ze and interacting with the quantum electromagnetic field can be written in the form (hereafter we put $\hbar = c = m = 1$)).

$$\left[Ze^2 - \tfrac{1}{2}rp^2 - r\sum_{ks} \omega_k C^+_{ks} C_{ks} + Er + \tfrac{1}{2}\delta m r p^2 \right.$$
$$\left. + e\sqrt{\frac{2\pi}{\Omega}} \sum_{ks} \frac{e^{(s)}_\lambda}{\sqrt{\omega_k}} (C_{ks} + C^+_{ks}) r p_\lambda \right] |\psi\rangle = 0 \tag{21}$$

In this formula $p_\lambda = -\mathrm{i}(\partial/\partial x_\lambda)$ $(\lambda = 1, 2, 3)$ and C^+_{ks}, C_{ks} are the operators of creation and annihilation of the photons with momentum $\boldsymbol{k}$, energy $\omega_k = |\boldsymbol{k}|$ and with the given state of polarisation; the unit vectors of polarisation satisfy the relations

$$e^{(s)}_\lambda e^{(s')}_\lambda = \delta_{ss'} \qquad \sum_s e^{(s)}_\lambda e^{(s)}_\mu = \delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2}$$

and Ω is the normalised volume. To eliminate the principal divergent terms we introduce the term of the mass renormalisation into equation (21) (see Bjorken and Drell 1964)

$$\delta m = \frac{8\pi e^2}{3\Omega} \sum_k \frac{1}{\omega_k^2}. \tag{22}$$

The description (21) of the interaction of the electron with the electromagnetic field in a 'dipole' approximation implies the upper bound on the energies of the photons under consideration.

The formal changes (see Komarov and Romanova 1982, 1985)

$$rp^2 \to -\frac{\partial^2}{\partial\xi^*_s \partial\xi_s} \qquad r \to \xi^*_s \xi_s \qquad x_\lambda \to (\sigma_\lambda)_{st} \xi^*_s \xi_t$$
$$rp_\lambda \to -\frac{\mathrm{i}}{2} (\sigma_\lambda)_{st} \left(\xi_t \frac{\partial}{\partial\xi_s} + \xi^*_s \frac{\partial}{\partial\xi^*_t} \right) \tag{23}$$

transform equation (21) to the equation describing interaction between a 'particle' with the complex coordinates $(s = 1, 2)$ and the quantum electromagnetic field. The scalar product of the wavefunctions in this two-dimensional complex space can be defined by the following relation:

$$\langle\varphi|\psi\rangle = \int \mathrm{d}^4\xi\, \varphi^*(\xi'_1, \xi''_1, \xi'_2, \xi''_2) \psi(\xi'_1, \xi''_1, \xi'_2, \xi''_2) \tag{24}$$

where $\xi'_s = \mathrm{Re}\ \xi_s$, $\xi''_s = \mathrm{Im}\ \xi_s$. In (21) all electron operators will henceforth be considered to be defined according to (23). The 'Hamilton operator' on the left-hand side of equation (21) is self-conjugate relative to the scalar product (24).

We shall take into account the interaction between the electron and the quantum electromagnetic field by perturbation theory assuming the charge of the electron e to be a small parameter. Then to a zeroth-order approximation, we have

$$|\psi_0\rangle = |0_f\rangle \psi_0(\xi) \tag{25}$$

where $|0_f\rangle$ is the vacuum state of the elctromagnetic field and the equation

$$(Ze^2-\tfrac{1}{2}rp^2-\tfrac{1}{2}\omega^2 r)\psi_0(\xi)=0 \tag{26}$$

defines the value of the parameter ω (we put $E_0=-\frac{1}{2}\omega^2$, i.e. we take one of the states of the hydrogen-like atom discrete spectrum as $\psi_0(\xi)$, δm in (21) is assumed to be a small value of order e^2). For the first-order correction to the wavefunction we obtain

$$|\psi_1\rangle=\sum_{ks} C_{ks}^{+}|0_f\rangle\varphi_{ks}(\xi) \tag{27}$$

where the functions $\varphi_{ks}(\xi)$ are defined by the equations

$$(Ze^2-\tfrac{1}{2}rp^2-\tfrac{1}{2}v_k^2 r)\varphi_{ks}(\xi)=-e\sqrt{\frac{2\pi}{\Omega\omega_k}}\,e_\lambda^{(s)}rp_\lambda\psi_0(\xi) \tag{28}$$

$$v_k^2=\omega^2+2\omega_k. \tag{29}$$

Solutions of the equations (28) can be represented in the form

$$\varphi_{ks}(\xi)=-\frac{e}{\mathrm{i}}\sqrt{\frac{2\pi}{\Omega\omega_k}}\,e_\lambda^{(s)}\lim_{\alpha\to 0}\int_0^\infty \mathrm{d}t \times\exp[-\alpha t+\mathrm{i}t(Ze^2-\tfrac{1}{2}rp^2-\tfrac{1}{2}v_k^2 r)]rp_\lambda\psi_0(\xi) \tag{30}$$

where we use an integral representation of an operator inverse to the self-conjugate operator

$$A^{-1}=\frac{1}{\mathrm{i}}\lim_{\alpha\to 0}\int_0^\infty \mathrm{d}t\exp(-\alpha t+\mathrm{i}At).$$

Note that

$$\int_0^\infty \mathrm{d}t\exp[\mathrm{i}t(Ze^2-\tfrac{1}{2}rp^2-\tfrac{1}{2}\omega^2 r)]\varphi(\xi)=\int \mathrm{d}^4\eta\, U(\xi,\eta;Ze^2)\varphi(\eta) \tag{31}$$

where the function $U(\xi,\eta;Ze^2)$ provides an 'energy' representation of the Green function. Finally, for the second-order correction to the energy of the system, we find the expression

$$E^{(2)}=\tfrac{1}{2}\delta m\frac{\langle\psi_0|rp^2|\psi_0\rangle}{\langle\psi_0|r|\psi_0\rangle}+\frac{1}{\langle\psi_0|r|\psi_0\rangle}\frac{2\pi e^2}{\mathrm{i}\Omega} \times\sum_{ks}\frac{e_\lambda^{(s)}e_\mu^{(s)}}{\omega_k}\lim_{\alpha\to 0}\int_0^\infty \mathrm{d}t\exp(\mathrm{i}tZe^2-\alpha t) \times\langle\psi_0|rp_\lambda\exp[-\mathrm{i}t(\tfrac{1}{2}rp^2+\tfrac{1}{2}v_k^2 r)]rp_\mu|\psi_0\rangle \tag{32}$$

where we use a short notation (24) for the scalar product of the functions. As is known, the formula (32) involves a shift of the level $\mathrm{Re}\,E^{(2)}$ and its radiative width $\Gamma=-2\,\mathrm{Im}\,E^{(2)}$.

Further calculations can be considerably simplified if instead of a direct use of the relation (31), we express the operators entering into (32) in terms of operators of creation (a_s^+, b_s^+) and annihilation (a_s, b_s) of the isotropic harmonic oscillator in

question. These operators are introduced through the relations (see Komarov and Romanova 1982):

$$a_s = \sqrt{\frac{\omega}{2}}\left(\xi_s + \frac{1}{\omega}\frac{\partial}{\partial \xi_s^*}\right) \qquad b_s = \sqrt{\frac{\omega}{2}}\left(\xi_s^* + \frac{1}{\omega}\frac{\partial}{\partial \xi_s}\right)$$
$$a_s^+ = \sqrt{\frac{\omega}{2}}\left(\xi_s^* - \frac{1}{\omega}\frac{\partial}{\partial \xi_s}\right) \qquad b_s^+ = \sqrt{\frac{\omega}{2}}\left(\xi_s - \frac{1}{\omega}\frac{\partial}{\partial \xi_s^*}\right) \tag{33}$$

and satisfy the commutation relations

$$[a_s, a_t^+] = \delta_{st} \qquad [b_s, b_t^+] = \delta_{st} \tag{34}$$

(the other commutators equal zero). It is convenient to utilise the value of the parameter ω from equation (26). Using (23) and (33) we find

$$rp^2 = \frac{\omega}{2}(2 + N - M - M^+) \qquad r = \frac{1}{2\omega}(2 + N + M + M^+)$$
$$rp_\lambda = -\frac{\mathrm{i}}{2}(m_\lambda - m_\lambda^+) \qquad x_\lambda = \frac{1}{2\omega}(m_\lambda + n_\lambda^a + n_\lambda^b + m_\lambda^+) \tag{35}$$

where

$$N = a_s^+ a_s + b_s^+ b_s \qquad M = a_s b_s \qquad M^+ = a_s^+ b_s^+$$
$$m_\lambda = (\sigma_\lambda)_{st} a_t b_s \qquad m_\lambda^+ = (\sigma_\lambda)_{st} a_s^+ b_t^+ \tag{36}$$
$$n_\lambda^a = (\sigma_\lambda)_{st} a_s^+ a_t \qquad n_\lambda^b = (\sigma_\lambda)_{st} b_t^+ b_s.$$

The operators (36) generate a closed algebra (see Komarov and Romanova 1985) permitting one to represent, for example, the operator

$$\exp[-\mathrm{i}t(\tfrac{1}{2}rp^2 + \tfrac{1}{2}\nu_k^2 r)]$$
$$= \exp\left(-\frac{\mathrm{i}t}{4}[\omega(2 + N - M - M^+) + \frac{\nu_k^2}{\omega}(2 + N + M + M^+)\right)$$

in the normal form

$$\exp[-\mathrm{i}t(\tfrac{1}{2}rp^2 + \tfrac{1}{2}\nu_k^2 r)]$$
$$= \mathrm{e}^{-\mathrm{i}\nu_k t}\left(\frac{4\nu_k\omega}{(\nu_k + \omega)^2 - (\nu_k - \omega)^2 \exp(-\mathrm{i}\nu_k t)}\right)^2$$
$$\times \exp\left(-\frac{(\nu_k^2 - \omega^2)[1 - \exp(-\mathrm{i}\nu_k t)]}{(\nu_k + \omega)^2 - (\nu_k - \omega)^2 \exp(-\mathrm{i}\nu_k t)} M^+\right)$$
$$\times \exp\left(-\frac{\mathrm{i}\nu_k t}{2} N + N \ln \frac{4\nu_k\omega}{(\nu_k + \omega)^2 - (\nu_k - \omega)^2 \exp(-\mathrm{i}\nu_k t)}\right)$$
$$\times \exp\left(-\frac{(\nu_k^2 - \omega^2)[1 - \exp(-\mathrm{i}\nu_k t)]}{(\nu_k + \omega)^2 - (\nu_k - \omega)^2 \exp(-\mathrm{i}\nu_k t)} M\right). \tag{37}$$

The use of (35) reduces equation (26) to the form

$$[Ze^2 - \tfrac{1}{2}\omega(2 + N)]|\psi_0\rangle = 0 \tag{38}$$

from which it follows that $|\psi_0\rangle$ is an eigenvector of the operator N. As shown by Komarov and Romanova (1982) those states are classified as 'physical' ones which belong to a zero eigenvalue of the operator

$$Q = a_s^+ a_s - b_s^+ b_s. \tag{39}$$

It is easy to see that any 'physical state' can be constructed through the action of the linear combinations of the powers of operators M^+ and m_λ^+ on the vacuum state $|0\rangle$ which is defined by the equations

$$a_s|0\rangle = b_s|0\rangle = 0 \tag{40}$$

(the classification of 'physical states' is presented in more detail by Komarov *et al* (1987)). If, for example, $|\psi_0\rangle$ is an arbitrary s state of the discrete spectrum of a hydrogen-like atom then there is a correspondence between this state and the vector

$$|\psi_0\rangle = \frac{1}{\sqrt{n!(n+1)!}}(M^+)^n|0\rangle \qquad n = 0, 1, 2, \ldots \tag{41}$$

and from (38) we find

$$\omega = \frac{Ze^2}{n+1} \tag{42}$$

($n+1$ is the principal quantum number according to the ordinary classification of states). Using the equations

$$N(M^+)^n|0\rangle = 2n(M^+)^n|0\rangle \qquad M(M^+)^n|0\rangle = n(n+1)(M^+)^{n-1}|0\rangle \tag{43}$$

obtained from the commutation relations

$$[M, M^+] = 2 + N \qquad [M, N] = 2M \qquad [N, M^+] = 2M^+ \tag{44}$$

we can easily find

$$\langle\psi_0|rp^2|\psi_0\rangle = \omega(n+1) \qquad \langle\psi_0|r|\psi_0\rangle = \frac{1}{\omega}(n+1)$$

and, finally,

$$\begin{aligned} E^{(2)} = {} & \tfrac{1}{2}\omega^2\delta m + \frac{\omega}{n+1}\frac{\pi e^2}{3\mathrm{i}\Omega}\int_0^\infty \mathrm{d}t\,\exp(\mathrm{i}tZe^2 - \mathrm{i}\nu_k t)[\alpha_k(t)]^2 \\ & \times \sum_{s=0}^{n} \frac{n!(n+1)!}{(s!)^2(n-s)!(n+3-s)!}\exp[-\mathrm{i}\nu_k t(n+1-s)][\alpha_k(t)]^{2(n+1-s)} \\ & \times [\beta_k(t)]^{2s}\left((n+3)(n+2) - \frac{1}{\beta_k^2(t)}s(s-1)\right)^2 \end{aligned} \tag{45}$$

where we introduce the definitions

$$\begin{aligned} \alpha_k(t) &= \frac{4\nu_k\omega}{(\nu_k+\omega)^2 - (\nu_k-\omega)^2\exp(-\mathrm{i}\nu_k t)} \\ \beta_k(t) &= \frac{(\nu_k^2-\omega^2)[1-\exp(-\mathrm{i}\nu_k t)]}{(\nu_k+\omega)^2 - (\nu_k-\omega)^2\exp(-\mathrm{i}\nu_k t)}. \end{aligned} \tag{46}$$

From a technical point of view it is an easy matter to further analyse equation (45) to extract the principal divergent (in the area of high energies of photons) terms and to obtain concrete numerical results.

In conclusion it may be said that the utilisation of the 'dipole' approximation for describing the interaction of an electron with a quantum electromagnetic field is totally unobligatory and the refusal of it does not actually increase the amount of algebra. In fact, in terms of the equation

$$\left[Ze^2 - \tfrac{1}{2}rp^2 - r\sum_{ks} \omega_k C_{ks}^+ C_{ks} + Er + \tfrac{1}{2}\delta m r p^2 + e\sqrt{\frac{2\pi}{\Omega}} \sum_{ks} \frac{e_\lambda^{(s)}}{\sqrt{\omega_k}} (C_{ks}\, e^{ikr} + C_{ks}^+ e^{-ikr}) rp_\lambda \right] |\psi\rangle = 0 \tag{47}$$

we obtain the following expression for the shift energy in a second approximation of perturbation theory:

$$\begin{aligned} E^{(2)} = \tfrac{1}{2}\delta m \frac{\langle\psi_0|rp^2|\psi_0\rangle}{\langle\psi_0|r|\psi_0\rangle} + \frac{1}{\langle\psi_0|r|\psi_0\rangle} \frac{2\pi e^2}{i\Omega} \\ \times \sum_{ks} \frac{e_\lambda^{(s)} e_\mu^{(s)}}{\omega_k} \int_0^\infty dt \exp(itZe^2) \\ \times \langle\psi_0|rp_\lambda\, e^{ikr} \exp[-it(\tfrac{1}{2}rp^2 + \tfrac{1}{2}\nu_k^2 r)]\, e^{-ikr} rp_\mu|\psi_0\rangle \end{aligned} \tag{48}$$

where now

$$\delta m = \frac{8\pi e^2}{3\Omega} \sum_k \frac{1}{\omega_k(\omega_k + \frac{1}{2}k^2)} \tag{49}$$

and the operators related to the electrons are, as before, understood in the sense of (23). Moving to the representation (33)–(35) and using the relations

$$\begin{aligned} e^{ikr} N\, e^{-ikr} &= N + \frac{i}{\omega} k_\lambda (m_\lambda - m_\lambda^+) + \frac{k^2}{2\omega^2}(2 + N + M + M^+) \\ e^{ikr}(M + M^+)\, e^{-ikr} &= M + M^+ - \frac{i}{\omega} k_\lambda (m_\lambda - m_\lambda^+) - \frac{k^2}{2\omega^2}(2 + N + M + M^+) \end{aligned} \tag{50}$$

which can readily be defined through the commutation relations for the operators (36) following from (34), we have

$$\begin{aligned} e^{ikr} \exp[-it(\tfrac{1}{2}rp^2 + \tfrac{1}{2}\nu_k^2 r)]\, e^{-ikr} \\ = \exp\left(-\frac{it}{4\omega} [2i\omega k_\lambda (m_\lambda - m_\lambda^+) + (\nu_k^2 + \omega^2 + k^2)(2 + N) \right. \\ \left. + (\nu_k^2 + k^2 - \omega^2)(M + M^+)] \right). \end{aligned} \tag{51}$$

Noticing now that the operators

$$\begin{aligned} C &= (\nu_k^2 - \omega^2 + k^2) M + 2i\omega k_\lambda m_\lambda) \\ C^+ &= (\nu_k^2 - \omega^2 + k^2) M^+ - 2i\omega k_\lambda m_\lambda^+ \end{aligned} \tag{52}$$

generate a closed algebra

$$\begin{aligned} [C, 2 + N] &= 2C \qquad [2 + N, C^+] = 2C^+ \\ [C, C^+] &= [(\nu_k^2 - \omega^2 + k^2)^2 + 4\omega^2](2 + N) \end{aligned} \tag{53}$$

we find a normal form of the operator (51)

$$
\begin{aligned}
&e^{ikr}\exp[-it(\tfrac{1}{2}rp^2+\tfrac{1}{2}\nu_k^2 r)]\,e^{-ikr} \\
&\quad=\exp(-i\nu_k t)\left(\frac{4\nu_k\omega}{(\nu_k+\omega)^2+k^2-[(\nu_k-\omega)^2+k^2]\exp(-i\nu_k t)}\right)^2 \\
&\qquad\times\exp\left(-\frac{1-\exp(-i\nu_k t)}{(\nu_k+\omega)^2+k^2-[(\nu_k-\omega)^2+k^2]\exp(-i\nu_k t)}\right. \\
&\qquad\left.\times[(\nu_k^2-\omega^2+k^2)M^{--}-2i\omega k_\lambda m_\lambda^+]\right) \\
&\qquad\times\exp\left(-\frac{i\nu_k t}{2}N+N\ln\frac{4\nu_k\omega}{(\nu_k+\omega)^2+k^2-[(\nu_k-\omega)^2+k^2]\exp(-i\nu_k t)}\right) \\
&\qquad\times\exp\left(-\frac{1-\exp(-i\nu_k t)}{(\nu_k+\omega)^2+k^2-[(\nu_k-\omega)^2+k^2]\exp(-i\nu_k t)}\right. \\
&\qquad\left.\times[(\nu_k^2-\omega^2+k^2)M+2i\omega k_\mu m_\mu]\right). \qquad (54)
\end{aligned}
$$

Using (54) makes the calculation of the matrix elements in (48) a relatively simple algebraic operation. Equations (48) and (49) do not have any divergences; therefore there is no need for the upper bound on the photon energy.

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